ON A THREE-DIMENSIONAL HEAT TRANSFER PROBLEM FOR A FORCED CONVECTIVE FLOW

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We consider the hydrodynamic and thermal boundary layers for the flow represented by Fig. 2. Assuming that $W \ll U_0$, we obtain approximate formulas for the friction and the heat transfer.

We consider a rotating tube 1 and inside of it a fixed tube 2, the latter containing a wide slot along a generator (Fig. 1); there is no gap between the tubes. A liquid of some kind is pumped through the tube, cooling the heated moving wall. If we disregard the secondary flows due to centrifugal forces and assume the boundary layer thicknesses to be small in comparison with the tube radius, then the motion represented by Fig. 2 can be regarded as an analog of the flow in question. Here the plate 1, which is infinite in the direction of the x axis and moving in that direction with speed U_0, is separated from the flow (of speed W) washing over it by the infinitely thin, fixed, thermally insulated plates 2.

We obtain an approximate solution of the hydrodynamic and thermal problems involved in determining the amount of heat given off to the flow by plate 1. The temperature T_0 of the moving plate is assumed to be known. Similar problems arise in the study of the sliding of bearings.

1. In the absence of longitudinal flows (W = 0) the hydrodynamic problem was considered in [1-3]. In particular, it was shown there that the speed at the outer edge of the boundary layer cannot be assigned arbitrarily but is determined from the equality of the frictional forces on the moving and the fixed sections:

$$U^0 = U_0\sqrt{s}, \quad s = l/L. \tag{1}$$

In view of the fact that the liquid is alternately speeded up by the moving plate and retarded by the fixed plate, the boundary layer turns out to be periodic in x of period L. The layer thickness δ_u is of the order

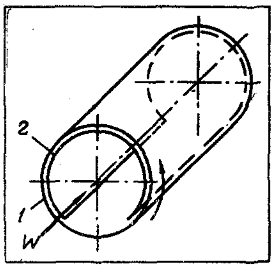

Fig. 1

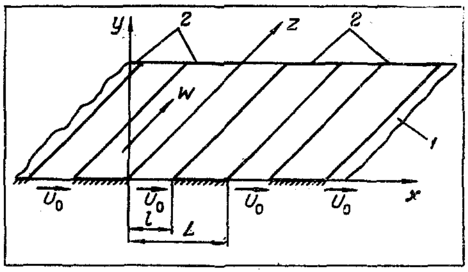

Fig. 2

Fig. 1. Schematic of liquid motion in the tube with a partially moving boundary.

Fig. 2. Schematic of flow over a plate consisting of fixed and moving sections.

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$$\delta_u \sim L\,\mathrm{Re}_x^{-0.5}, \quad \mathrm{Re}_x = U_0 L/\nu. \tag{2}$$

If $W \neq 0$, but $W \ll U^0$, then the boundary layer can be divided into two zones. In the first zone, situated near the wall, the influence of the longitudinal flow is small, so that direct use can be made of the results obtained in [1-3]. The thickness of this zone is of order δ_u. In the second zone the speed w increases gradually to its limiting value W as $y \to \infty$, while u decreases from U^0 to $u = 0$ (we assume that $u = 0$ for $z < 0$). By virtue of the assumption $W \ll U^0$, we have, over the major part of the boundary layer,

$$\varepsilon = \delta_u/\delta \ll 1. \tag{3}$$

The speeds $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ in the second zone satisfy the equations

$$\tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \tilde{u}}{\partial y} + \tilde{w}\frac{\partial \tilde{u}}{\partial z} = \nu\frac{\partial^2 \tilde{u}}{\partial y^2}, \quad \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = 0,$$
$$\tilde{u}\frac{\partial \tilde{w}}{\partial x} + \tilde{v}\frac{\partial \tilde{w}}{\partial y} + \tilde{w}\frac{\partial \tilde{w}}{\partial z} = 0, \tag{4}$$

and the boundary conditions can then be written approximately as follows:

$$\tilde{u} = U^0, \quad \tilde{w} = 0, \quad \tilde{v} = v_\varepsilon(x) \quad \text{for } y \to 0, \quad \tilde{u} = 0, \quad \tilde{w} = W \quad \text{for } y \to \infty. \tag{5}$$

Here v_ε is the v component at the outer edge of the first zone, where the integral of v_ε with respect to x is equal to zero over an arbitrary interval of length L.

We can show that

$$\tilde{u}/U^0 = 1 - \tilde{w}/W = \omega(x, y, z). \tag{6}$$

Applying the method of integral relations to Eqs. (4) and putting

$$\omega = 1 - 1.5\eta + 0.5\eta^3, \quad \eta = y/\delta, \tag{7}$$

for the boundary layer thickness in the second zone, we obtain

$$\frac{\partial \delta}{\partial z} + \frac{q_1 U^0}{W}\cdot\frac{\partial \delta}{\partial x} = \frac{q_2 \nu}{\delta W} + \frac{2 q_2 v_\varepsilon}{3W}, \quad q_1 = \frac{36}{13}, \quad q_2 = \frac{140}{13}. \tag{8}$$

Along the integral curves of the equation

$$dz = (W/q_1 U^0)\,dx \tag{9}$$

the quantity δ, by virtue of Eqs. (8), satisfies the equation

$$\frac{d\delta}{dz} = \frac{q_2 \nu}{\delta W} + \frac{2 q_2 v_\varepsilon}{W}. \tag{10}$$

We consider a function $\delta^0(z)$ such that

$$\frac{d\delta^0}{dz} = \frac{q_2 \nu}{\delta^0 W}, \quad \delta^0 = \sqrt{\delta_0^2 + \frac{2 q_2 \nu}{W}(z - z_0)}, \quad \delta_0 = \delta(z = z_0), \tag{11}$$

and we seek δ in the form $\delta = \delta^0 + \Delta$.

Taking into account the aforementioned properties of the flow in the first zone, we can show that $\Delta \sim \delta_u$ and, by virtue of relation (3), that $\delta \approx \delta^0$. The latter, generally speaking, is valid at some distance from the entrance edge of the plate where the condition (3) is already satisfied. If, however, we assume that $z \gg z_0$, or, equivalently, if we take $z_0 = 0$, $\delta_0 = 0$, then Eq. (11) for δ can be approximately extended to zero values of z and may be written in the form

$$\delta = \sqrt{\frac{2 q_2 \nu z}{W}}. \tag{12}$$

Equation (12) agrees with the expression given in [4] for the boundary layer thickness in the absence of flow along the x axis; thus it shows, under the assumptions made above, that the hydraulic resistance along the z axis and the profile of the velocity w can be determined for the boundary layer on the plate from known formulas.

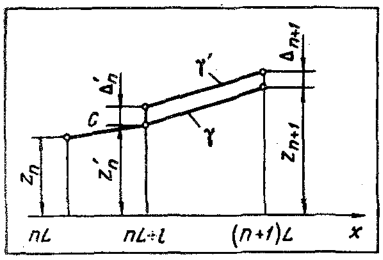

Fig. 3. Integral curves of Eqs. (23) and (19).

2. Consider now the thermal part of the problem. The heat balance equation may be written as follows:

$$\frac{\partial}{\partial x}(uT)+\frac{\partial}{\partial y}(vT)+\frac{\partial}{\partial z}(wT)=\frac{\lambda}{\rho c}\frac{\partial^2 T}{\partial y^2}. \tag{13}$$

The boundary conditions (n is an integer) are

$$T=0 \text{ for } y\to\infty,\quad T=T_0 \text{ for } y=0,\quad nL<x<nL+l,$$
$$\partial T/\partial y=0 \text{ for } nL+l<x<(n+1)L. \tag{14}$$

Further, we consider liquids with Prandtl numbers $\Pr = \nu\rho c/\lambda \gg 1$, so that the thermal boundary layer thickness δ_t is small compared to δ_u and δ. Then, approximately, over the major part of the boundary layer, for $0 < y < \delta_t$

$$u\approx U_0,\quad w\approx k_w y \quad \text{for} \quad nL<x<nL+l,$$
$$u\approx k_u y,\ w\approx k_w y \quad \text{for} \quad nL+l<x<(n+1)L, \tag{15}$$
$$k_w=1.5W/\delta,\quad k_u=\tau_{xy}(y=0)/\mu.$$

We can show that, approximately,

$$\tau_{xy}(y=0)\approx\frac{\mu U_0^{3/2}}{(\nu L)^{1/2}}\chi,\quad \chi=\frac{1}{\pi^{3/2}}\cdot\frac{s^{-1/4}}{1-s}\sum_{k=1}^{\infty}\frac{\sin^2\pi ks}{k\sqrt{k}}. \tag{16}$$

Consider the segment $nL < x < nL + l$. Noting that $v(y = 0) = 0$, and taking the following dependence for the temperature

$$T=T_0\vartheta(\eta),\quad \eta=y/\delta_t,\quad \vartheta=1-1.5\eta+0.5\eta^3, \tag{17}$$

we obtain, after integrating Eq. (13) with respect to y,

$$\beta\frac{\partial\delta_t}{\partial x}+\frac{\partial\delta_t}{\partial z}=\frac{1}{2}\cdot\frac{\delta_t}{\delta}\cdot\frac{\partial\delta}{\partial z}-\frac{\vartheta'(0)}{3a_1}\cdot\frac{\nu}{\Pr}\cdot\frac{1}{W}\cdot\frac{\delta}{\delta_t^2},$$

$$\beta=\frac{a_0}{3a_1}\cdot\frac{U_0}{W}\frac{\delta}{\delta_t},\quad a_0=\int_0^1\vartheta\,d\eta,\quad a_1=\int_0^1\eta\vartheta\,d\eta. \tag{18}$$

Along the integral curves of the equation

$$dx=\beta dz \tag{19}$$

we have, by virtue of relations (18),

$$\frac{d\psi}{dz}+\frac{3}{4}\cdot\frac{\psi}{z}=\frac{3}{4z},\quad \psi=\frac{14}{13}\left(\frac{\delta_t}{\delta}\right)^3\Pr. \tag{20}$$

Assume that at some point $x = nL + 0$, $z = z_n$ (see Fig. 3) the quantity $\delta_t = \delta_t$ is known. Integrating Eq. (20), we find that along the integral curve C of Eq. (19) passing through this point

$$\psi=1-(1-\psi_n)(z_n/z)^{3/4}, \tag{21}$$

and, at the point $x = nL + l-0$, $z = z_n'$ of the curve C, we have

$$\psi_n'=1-(1-\psi_n)(z_n/z_n')^{3/4}. \tag{22}$$

Through the point $x = nL + l$, $z = z_n'$, and the point $x = nL + l$, $z = z_n' + \Delta_n'$ infinitely close to it, we draw integral curves γ and γ' of the equation

$$k_w dx=k_u dz. \tag{23}$$

On the curves γ and γ' as guides we construct cylindrical surfaces whose generators are perpendicular to the plane $y = 0$. These surfaces, together with the planes $x = nL + l$, $x = (n + 1)L$, $y = 0$, and the surface $y = \delta_t$, form a volume V. We integrate Eq. (13) over V and apply the Ostrogradskii–Gauss formula of the resulting left-hand side. Taking into account the fact that $u \approx U_0 = \text{const}$ for $nL < x < nL + l$, we can obtain, to within a quantity of order ε^3,

$$\delta_{t_{n+1}} = \delta_{t_n}, \quad \delta'_{t_n} = \delta_t\Big|_{\substack{x=nL+l-0\\ z=z'_n}}, \quad \delta_{t_{n+1}} = \delta_t\Big|_{\substack{x=(n+1)L+0 \\ z=z_{n+1}}}. \tag{24}$$

From Eqs. (22) and (24), with the aid of Eqs. (20) and (12), we find a recursion relation for determining ψ_n:

$$\psi_{n+1} = (z_n/z_{n+1})^{3/2}\left[1 - (1-\psi_n)(z_n/z'_n)^{3/4}\right]. \tag{25}$$

Integrating Eqs. (19) and (23), we can show that

$$\frac{z'_n - z_n}{z_n} = \left(\frac{13}{14}\right)^{1/3} \frac{3a_1 s}{a_0} \cdot \frac{WL}{U_0 z_n}\left(\frac{\psi_n}{\Pr}\right)^{1/3}\left[1 + O(\varepsilon^3)\right] \sim \psi_n \varepsilon^3,$$

$$\frac{z_{n+1} - z'_n}{z_n} = \frac{2}{3}\, \frac{(1-s)\sqrt{2q_2}}{\chi}\left(\frac{WL}{U_0 z_n}\right)^{3/2}\left[1 + O(\varepsilon^3)\right] \sim \varepsilon^3. \tag{26}$$

An analysis of Eqs. (25) and (26) shows that the function ψ_n, which satisfies the recursion relation (25), differs by no more than a quantity of order ε^3 from the solution of the equation

$$\frac{d\bar\psi}{dz} = \frac{3}{4} \cdot \frac{1+\xi}{z}\left(\frac{1-\xi}{1+\xi} - \bar\psi\right), \quad \xi = (1 + 3\varkappa\bar\psi^{1/3})^{-1},$$

$$\varkappa = \frac{0.807s}{1-s}\left(\frac{U_0 z}{WL}\right)^{1/2} \frac{1}{\Pr^{1/3}}, \quad \varkappa\psi_n^{1/3} \sim \frac{\delta_t}{\delta_u}. \tag{27}$$

obtained using the same initial conditions.

Equation (27) possesses the property that sufficiently large z/z_0 its solution does not depend on $\psi_0 = \bar\psi(z_0)$. Formulas for ψ_n, similar to Eq. (12), obtained by the method indicated here, are valid for those $z \geq z_0$ for which the inequality (3) is already satisfied. Further, as in the hydrodynamic part of the problem, we consider values of $z \gg z_0$.

Assuming, as we did earlier, that the quantity δ_t/δ_u is small, we obtain an approximate solution of Eq. (27), which is independent of the initial conditions, in the form

$$\bar\psi \approx \varkappa^{3/2}(z). \tag{28}$$

It follows from this, in particular, that $3\varkappa\bar\psi^{1/3} \approx 3\varkappa^{3/2}$, so that the solution obtained is valid as long as the quantity $\varkappa$ is small.

The solution (28) may be used to determine $\psi(z)$ for all x in the intervals $nL < x < nL + l$. The latter is ensured by the continuity of $\psi(z)$ from Eq. (21) and the smallness of the increment $\Delta z'$ (see Eqs. (26) on each interval $nL < x < nL + l$ of the integral curve of Eq. (19).

3. We calculate the heat-transfer coefficient from the moving plate. Determining δ_t on the interval $nL < x < nL + l$ with the aid of Eqs. (28), (27), (20), and (12), and using the relations (17), for the local Nusselt number $\mathrm{Nu} = \alpha z/\lambda = 3z/2\delta_t$ we find

$$\mathrm{Nu} = \varkappa^{-1/2}\mathrm{Nu}_0, \quad \mathrm{Nu}_0 = 0.332\mathrm{Re}_z^{1/2}\Pr^{1/3}, \quad \mathrm{Re}_z = \frac{Wz}{\nu}. \tag{29}$$

The formula (29) is not suitable if $\delta_t > \delta_u$, i.e., if the quantity $\varkappa$ is not small. In this case, at least for $\delta_t \gg \delta_u$, we can obtain, for large Prandtl numbers Pr, with the aid of reasoning to a certain extent analogous to the preceding,

$$\mathrm{Nu} = s^{-1/3}\mathrm{Nu}_0. \tag{30}$$

Equations (29) and (30) show that in comparison with the flow over the fixed plate ($\mathrm{Nu} = \mathrm{Nu}_0$, [4]) the heat transfer, under the conditions of the arrangement shown in Fig. 2, turns out to be more intensive. This is explained by the fact that in the latter case the speed in the boundary layer has, besides that along the z axis, also a component along the x axis. Therefore the coolant liquid, passing not only through a heat emitting region but also a thermally insulated region, is heated on only a portion of the path which it traverses. As a result there is a decrease in the thermal boundary layer thickness, and, hence, an increase in the heat-transfer coefficient.

NOTATION

U_0	is the velocity of movable plate;
U^0	is the velocity at outer edge of boundary layer without longitudinal flows;
W	is the velocity of longitudinal flow;
s	is the relative length of movable plate;
δ_u, δ	are the hydrodynamic boundary layers thicknesses;
$\varepsilon = \delta_u/\delta$;	
δ_t	is the thermal boundary layer thickness;
T	is the temperature;
μ, ν	are the dynamic and kinematic viscosities;
ψ	is the value by formula (20);
$\varkappa$	is the value by formula (27);
α	is the heat-transfer coefficient of movable wall;
Nu	is the Nusselt number.

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